Observer-based relay feedback controller design for LTI systems

Zohra Kader, Christophe Fiter, Laurentiu Hetel, Lotfi Belkoura

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Abstract— **This paper presents a design approach for observer-based relay feedback controllers. A switching law dependent on the estimation state is designed while using a Luenberger observer. The stabilization problem leads to qualitative conditions. A numerical example is provided to assess the effectiveness of the developed method.**

I. INTRODUCTION

Relays are largely studied in control theory since the fifties [8], [27]. They are widely used in different application fields and for different targets - see for instance [13], [18], [29], [30]. Relay feedback controllers present some advantages which make them a perfect substitute to continuous control laws (see for instance [11], [14], [27], [28]). Indeed, they can emulate locally the behavior of a linear static feedback, and they are classified as simple and robust controller [7], [14], [23]. However, the usefulness of relays for stabilization and control does not exclude difficulties and some undesired phenomena. From a theoretical point of view, systems with relay feedback control can be seen as switched systems [17] with a complex behaviour. The design of a relay feedback controller is not an obvious problem even for the case of linear systems. In the literature [15], [16], the presence of sliding modes, limit cycles and chattering in relay feedback systems is pointed out. These phenomena must not be neglected and their study is theoretically challenging. In particular, for systems with sliding modes the notion of system's solution must be reviewed to take into account the dynamics obtained by fast switching [5], [9]. Frequency domain methods [3] and LMI approaches [22], [23] have also been used for relay feedback controller design. Recently, a convex embedding formalism has been used in order to design relay feedback controllers in [14] and [12]. However, to the best of our knowledge, the existing results about relay feedback systems consider the system's states as perfectly known. In many practical cases, the state of the system is not fully available for measurements. In this case an observer-based controller must be designed.

Here, we design a relay feedback controller with an observer-based switching law. Using a convex embedding formalism [6], [14], it will be shown how we can design an observer and switching hyperplane so as to ensure the local exponential stability of the closed loop system. The research here is also related with the output feedback sliding mode control problem [21], [10] and output feedback design for switched systems [25].

The paper is organized as follows: Section II gives the system description and exposes the problem under study. A qualitative stability result is proposed in Section III. In Section IV, a numerical example is given to illustrate the efficiency of the presented method. Finally, perspectives are given in the last section together with the conclusion.

A. Notations

In this paper we use the notation $\mathbb{R}^+$ to refer to the interval $[0, \infty)$. The transpose of a matrix M is denoted by M^T and if the matrix is symmetric the symmetric elements are denoted by $*$. The notation $M \succeq 0$ (resp. $M \preceq 0$) means that the matrix M is positive (resp. negative) semi-definite, and the notation $M \succ 0$ (resp. $M \prec 0$) means that it is positive definite (resp. negative definite). The identity matrix is denoted by I and both notations $\text{eig}_{min}(M)$ and $\text{eig}_{max}(M)$ are used to refer to the minimum and maximum eigenvalue respectively of a matrix M. For a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a positive scalar γ, we denote by $\mathcal{E}(P, \gamma)$ the ellipsoid

$$\mathcal{E}(P, \gamma) = \{x \in \mathbb{R}^n : x^T P x \leq \gamma\}, \tag{1}$$

and for all positive scalar r.

For a given set $\mathcal{S}$, the notation $\text{Conv}\{\mathcal{S}\}$ indicates the convex hull of the set, $\text{int}\{\mathcal{S}\}$ its interior and $\overline{\mathcal{S}}$ its closure and finally the closed convex hull of the set $\mathcal{S}$ will be noted by $\overline{\text{Conv}}\{\mathcal{S}\}$. The minimum argument of a given function $f : \mathcal{S} \longrightarrow \mathbb{R}$ such that the set $\mathcal{S} \subset \mathbb{R}$ is a finite set of vectors is noted by

$$\arg\min f = \{y \in \mathcal{S} : f(y) \leq f(z), \forall z \in \mathcal{S}\}. \tag{2}$$

For a positive integer N, we denote by $\mathcal{I}_N$ the set $\{1, \ldots, N\}$. By Δ_N we denote the unit simplex

$$\Delta_N = \left\{ \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0, i \in \mathcal{I}_N \right\}. \tag{3}$$

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II. System description and problem statement

A. System description

Consider the linear system

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{4}$$

with $x \in \mathbb{R}^n$, an input u which takes values in the set $\mathcal{V} = \{v_1, \ldots, v_N\} \subset \mathbb{R}^m$ and an output $y \in \mathbb{R}^p$. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are the matrices describing the system.

In the sequel we assume that:

- A-1 The pair (A, B) is stabilizable, which means that there exists a matrix K such that the closed-loop matrix $A_{cl} = A + BK$ is Hurwitz.
- A-2 The set $\text{int}\{\text{Conv}\{\mathcal{V}\}\}$ is nonempty and the null vector is contained inside ($0 \in \text{int}\{\text{Conv}\{\mathcal{V}\}\}$).
- A-3 The pair (A, C) is detectable, which means that there exists a matrix L such that the matrix $A_o = A + LC$ is Hurwitz.

This paper deals with the stabilization of system (4) in the case of an observer-based switching law. We consider a controller given by

$$u(\hat{x}) \in \arg\min_{v\in\mathcal{V}} \hat{x}^T \Gamma v, \tag{5}$$

where the matrix $\Gamma \in \mathbb{R}^{n\times m}$ characterizes the switching hyperplanes, and $\hat{x} \in \mathbb{R}^n$ is the estimated state which is computed by the full-order Luenberger state observer [19], [20]

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(\hat{y} - y), \\ \hat{y} = C\hat{x}. \end{cases} \tag{6}$$

The formulation of the controller (5) encompasses the classical sign function in the classical relay feedback. Note that, if $\mathcal{V} = \{v_1, v_2\} = \{-v, v\}$ with $v > 0$ then, we get

$$\begin{aligned} u(\hat{x}) &= -v\text{sign}(\Gamma\hat{x}) \\ &\in \begin{cases} v & \text{if} \quad \Gamma\hat{x} < 0, \\ \{-v, v\} & \text{if} \quad \Gamma\hat{x} = 0, \\ v & \text{if} \quad \Gamma\hat{x} > 0. \end{cases} \end{aligned} \tag{7}$$

Our objective is to provide conditions which guarantee the existence of matrices Γ (which characterizes the switching hyperplanes of the control law) and L (the observer gain) such that the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ LC & A+LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u \tag{8}$$

with the control law (5) is locally exponentially stable (this problem will be mathematically formalized farther in II-C).

B. Solution concept

Using the augmented state

$$\xi = \begin{bmatrix} \hat{x} \\ e \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \tag{9}$$

where $e = \hat{x} - x$ is the estimation error, the interconnection (4), (6) can be written as the augmented closed-loop system

$$\begin{cases} \dot{\xi} &= \begin{bmatrix} A & LC \\ 0 & A+LC \end{bmatrix} \xi + \begin{bmatrix} B \\ 0 \end{bmatrix} u(\hat{x}), \\ y &= \begin{bmatrix} C & C \end{bmatrix} \xi, \end{cases} \tag{10}$$

which leads to

$$\begin{cases} \dot{\xi} &= \tilde{A}\xi + \tilde{B}\bar{u}(\xi) = \mathcal{X}(\xi), \\ y &= \begin{bmatrix} C & C \end{bmatrix} \xi, \end{cases} \tag{11}$$

where $\tilde{A} = \begin{bmatrix} A & LC \\ 0 & A+LC \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, and

$$\bar{u}(\xi) = u(\begin{bmatrix} I & 0 \end{bmatrix} \xi) = u(\hat{x}) \in \arg\min_{v\in\mathcal{V}} \xi^T \begin{bmatrix} I \\ 0 \end{bmatrix} \Gamma v. \tag{12}$$

Note that this is a differential equation with a discontinuous right hand side [9], [5], and thus we need an appropriate formalism and specific tools to define the system's solutions and analyze their behaviour.

Therefore, to the discontinuous closed-loop system (11), (12) we associate the differential inclusion

$$\dot{\xi} \in \mathcal{F}[\mathcal{X}](\xi), \tag{13}$$

with $\mathcal{F}[\mathcal{X}](\xi)$ the set-valued map which can be computed from the differential equation with a discontinuous right hand side using the construction given in [2], [5], [9], [24]

$$\mathcal{F}[\mathcal{X}](\xi) = \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \overline{\text{Conv}}\{\mathcal{X}(\mathcal{B}(\xi,\delta)) \setminus \mathcal{S}\}, \xi \in \mathbb{R}^{2n}, \tag{14}$$

where $\overline{\text{Conv}}$ is the closed convex hull, $\mathcal{B}(\xi, \delta)$ is the open ball centered on ξ with radius δ, and $\mathcal{S}$ is a set of measure zero with $\mu(\mathcal{S})$ its measure in the sense of Lebesgue. The closed-loop system is then modeled by a differential inclusion for which the notion of a solution was defined in [9], and recalled hereafter.

Definition 1: (Filippov solution) Consider the closed-loop system (11) and its associated differential inclusion (13). A *Filippov solution* of the discontinuous system (11), (12) over the interval $[t_a, t_b] \subset [0, \infty)$ is an absolutely continuous mapping $y(t) : [t_a, t_b] \longrightarrow \mathbb{R}^{2n}$ satisfying:

$$\dot{y}(t) \in \mathcal{F}[\mathcal{X}](y(t)), \quad \text{for almost all} \quad t \in [t_a, t_b], \tag{15}$$

with $\mathcal{F}[\mathcal{X}](\xi)$ given by (14).

A differential inclusion has at least one solution if the set valued map $\mathcal{F}[\mathcal{X}](\xi)$ is locally bounded and takes nonempty, compact and convex values [1], [2], [5], [9] which is the case of the differential inclusion (13) corresponding to the system (11), (12).

C. Problem statement

Hereafter the notion of stability which will be used is introduced and we mathematically formalize the problem under study.

Definition 2: (local exponential stability) The differential inclusion (13) is said to be *locally exponentially stable* with a decay rate α (it is also said to be locally α-stable) to the origin in a compact set Ω containing the origin if there exist positive scalars c and α such that every possible solution $\xi(t)$ of (13) starting from any initial condition $\xi(0) \in \Omega$ verifies

$$\|\xi(t)\| \leq ce^{-\alpha t}\|\xi(0)\|. \tag{16}$$

We recall that sufficient conditions for the local exponential stability with decay rate α of a differential equation with a discontinuous right hand side $\dot{\xi} = \mathcal{X}(\xi)$, with $\mathcal{X}$ locally bounded, are given by the existence of a strict Lyapunov function V, $V(0) = 0$, $V(\xi) > 0$, $\forall \xi \neq 0$, such that

$$\sup_{y \in \mathcal{F}[\mathcal{X}](\xi)} \frac{\partial V}{\partial \xi} y \leq -2\alpha V(\xi), \forall \xi \in \mathcal{D}, \tag{17}$$

for some positive scalar α and a domain $\mathcal{D}$ such that $0 \in \text{int}\{\mathcal{D}\}$.

The main problem under study is mathematically formulated as follows:

Problem. Are there matrices Γ and L such that system (11)-(12) is locally exponentially stable (when solutions are understood in the sense of Filippov)?

III. Observer-based control design

This section deals with the local exponential stabilization of system (11), (12) and equivalently with the local exponential stabilization of system (4), (6) by the switching law (5). Assumptions A.1, A.2 and A.3 are used to prove that there exist a switching matrix Γ and an observer gain L such that the system is locally exponentially stable. The results are given in the following.

Theorem 1: Assume that A.1, A.2 and A.3 hold. Then there exist matrices Γ (characterizing the switching hyperplanes) and L (the observer gain) such that system (11), (12) (or equivalently the closed-loop system (4), (5), (6)) is locally exponentially stable in an ellipsoidal domain containing the origin.

Proof: Since system (4) is stabilizable, then there exist a static gain K, a scalar $\alpha_K > 0$ and a symmetric positive definite matrix P_1 such that $A_{cl} = A + BK$ is Hurwitz and satisfies

$$A_{cl}^T P_1 + P_1 A_{cl} \preceq -2\alpha_K P_1. \tag{18}$$

Likewise, since the system is observable, then there exist an observer gain L, a scalar $\alpha_o > 0$ and a symmetric positive definite matrix P_2 such that $A_o = A + LC$ is Hurwitz and satisfies

$$A_o^T P_2 + P_2 A_o \preceq -2\alpha_o P_2. \tag{19}$$

We want to prove that the system (11), (12) is locally exponentially stable in some domain $\mathcal{D}$ around the origin. Let us consider the quadratic Lyapunov function

$$V(\xi) = \xi^T P \xi \tag{20}$$

with the $2n \times 2n$ matrix

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & \lambda P_2 \end{bmatrix} \tag{21}$$

with a scaling term $\lambda > 0$. We want to show then that taking the matrix Γ defined in (5) as

$$\Gamma = P_1 B = \begin{bmatrix} I & 0 \end{bmatrix} P\tilde{B} \tag{22}$$

with $\tilde{B}$ defined in (12) and for some positive scalar α we have

$$\sup_{y \in \mathcal{F}[\mathcal{X}](\xi)} \frac{\partial V}{\partial \xi} y \leq -2\alpha V(\xi), \tag{23}$$

in a domain $\mathcal{D} \subset \mathbb{R}^{2n}$ to be determined.

For each $\hat{x} \in \mathbb{R}^n$ we define the set of minimizers in which the control (5) takes values. This corresponds to defining minimizers in which the control (12) takes values such that

$$\hat{x}^T \Gamma v = \hat{x}^T P_1 B v = \xi^T \begin{bmatrix} I \\ 0 \end{bmatrix} P\tilde{B}v. \tag{24}$$

We define for any $z \in \mathbb{R}^{2n}$ the set of indexes $\mathcal{I}^*(z)$ such that

$$\mathcal{I}^*(z) = \left\{ i \in \mathcal{I}_N : z^T P\tilde{B}(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N \right\}, \tag{25}$$

with $\tilde{B}$ defined in (11). To $\mathcal{I}^*(z)$ we associate for all $z \in \mathbb{R}^{2n}$ the set $\Delta^*(z)$ of vectors defined by:

$$\Delta^*(z) = \left\{ \beta \in \Delta_N : \beta_i = 0, \forall i \in \mathcal{I}_N \setminus \mathcal{I}^*(z) \right\}. \tag{26}$$

Using (25) and (26), the set valued map $\mathcal{F}[\mathcal{X}](\xi)$ in (13) satisfies

$$\mathcal{F}[\mathcal{X}](\xi) \subseteq \mathcal{F}^*[\mathcal{X}](\xi) \tag{27}$$

with

$$\begin{aligned} \mathcal{F}^*[\mathcal{X}](\xi) &= \overline{\text{Conv}}_{i \in \mathcal{I}^*(\xi)} \{\tilde{A}\xi + \tilde{B}v_i\} \\ &= \{\tilde{A}\xi + \tilde{B}v(\beta) : \beta \in \Delta^*(\xi)\}, \end{aligned} \tag{28}$$

with $v(\beta) = \sum_{i=1}^N \beta_i v_i$.

Consider the gain K satisfying (18). From (27) and (28) and using the fact that $\Delta^*(\xi)$ is compact, we have

$$\begin{aligned} \sup_{y \in \mathcal{F}[\mathcal{X}](\xi)} \frac{\partial V}{\partial \xi} y &\leq \sup_{y \in \mathcal{F}^*[\mathcal{X}](\xi)} \frac{\partial V}{\partial \xi} y \\ &= \sup_{\beta \in \Delta^*(\xi)} \left\{ \frac{\partial V}{\partial \xi} \left\{ \tilde{A}\xi + \tilde{B}v(\beta) \right\} \right\} \\ &= \max_{\beta \in \Delta^*(\xi)} \left\{ \frac{\partial V}{\partial \xi} \left\{ \tilde{A}\xi + \tilde{B}v(\beta) \right\} \right\}. \end{aligned} \tag{29}$$

Thus, in order to show (23), it is sufficient to prove that for some scalar $\alpha > 0$ we have

$$\max_{\beta \in \Delta^*(\xi)} \left\{ \frac{\partial V}{\partial \xi} \left\{ \tilde{A}\xi + \tilde{B}v(\beta) \right\} \right\} \leq -2\alpha V(\xi), \tag{30}$$

in a domain $\mathcal{D}$ to be determined.

Note that, since Assumption A-2 holds, then there exists a neighborhood of the origin $\mathcal{E}(P, \gamma) \subset \mathbb{R}^{2n}$, with $\gamma > 0$ such that for all $\xi = \begin{bmatrix} \hat{x} \\ e \end{bmatrix} \in \mathcal{E}(P, \gamma)$, we have

$$K\hat{x} = \mathcal{K}\xi \in \text{Conv}\{\mathcal{V}\}, \tag{31}$$

with $\mathcal{K} = \begin{bmatrix} K & 0 \end{bmatrix}$.

Therefore, for all $\xi \in \mathcal{E}(P, \gamma)$ there exist scalars $\alpha_j(\xi)$, $j \in \mathcal{I}_N$ such that $\sum_{j=1}^N \alpha_j(\xi) = 1$ and

$$\mathcal{K}\xi = \sum_{j=1}^N \alpha_j(\xi) v_j. \tag{32}$$

From (25), for all $i \in \mathcal{I}^*(\xi)$ we have

$$\xi^T P\tilde{B}(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N. \tag{33}$$

Then, for any $\beta \in \Delta^*(\xi)$, we have

$$\xi^T P\tilde{B}(v_j - v(\beta)) \geq 0, \forall j \in \mathcal{I}_N. \tag{34}$$

Then, considering (32), and multiplying the last inequalities by $\alpha_j(\xi)$ and summing the N elements we obtain

$$\xi^T P\tilde{B}(\mathcal{K}\xi - v(\beta)) \geq 0. \tag{35}$$

Adding this to the left part of (30), it comes

$$\begin{aligned}
&\max_{\beta\in\Delta^*(\xi)} \left\{ \frac{\partial V}{\partial \xi} \left\{ \tilde{A}\xi + \tilde{B}v(\beta) \right\} \right\} \\
&\leq \max_{\beta\in\Delta^*(\xi)} \left\{ 2\xi^T P\{\tilde{A}\xi + \tilde{B}v(\beta)\} \right\} + 2\xi^T P\tilde{B}(\mathcal{K}\xi - v(\beta)) \\
&= 2\frac{\partial V}{\partial \xi} \begin{bmatrix} A_{cl} & LC \\ 0 & A_o \end{bmatrix} \xi = 2\frac{\partial V}{\partial \xi}(\tilde{A}_{cl}\xi) = 2\xi^T P(\tilde{A}_{cl}\xi).
\end{aligned} \tag{36}$$

Thus, in order to show (23), it is sufficient to prove that

$$2\xi^T P(\tilde{A}_{cl}\xi) \leq -2\alpha V(\xi), \forall \xi \in \mathcal{E}(P, \gamma), \tag{37}$$

which holds if

$$\tilde{A}_{cl}^T P + P\tilde{A}_{cl} \preceq -2\alpha P. \tag{38}$$

Note that

$$\begin{aligned}
&\tilde{A}_{cl}^T P + P\tilde{A}_{cl} + 2\alpha P = \\
&\begin{bmatrix} A_{cl}^T P_1 + P_1 A_{cl} + 2\alpha P_1 & P_1 LC \\ (LC)^T P_1 & \lambda(A_o^T P_2 + P_2 A_o + 2\alpha P_2) \end{bmatrix}.
\end{aligned} \tag{39}$$

Applying the Schur complement, the matrix (39) is negative if and only if

$$A_o^T P_2 + P_2 A_o + 2\alpha P_2 \preceq 0 \tag{40}$$

and

$$\begin{aligned}
&(A_{cl}^T P_1 + P_1 A_{cl} + 2\alpha P_1) \\
&-\frac{1}{\lambda} P_1 LC[2\alpha P_2 + A_o^T P_2 + P_2 A_o]^{-1}(LC)^T P_1 \\
&\preceq 0.
\end{aligned} \tag{41}$$

Since (18) and (19) are satisfied, it is obvious that if we take $\alpha \leq \min(\alpha_K, \alpha_o)$, and λ large enough both inequalities are verified.

Thus, there exist $\Gamma = P_1 B$ and an observer gain L such that system (11), (12) (and equivalently (4), (5), (6)) is locally α-stable with a domain of attraction $\mathcal{E}(P, \gamma)$. ■

Remark 1: Note that, the proof of Theorem 1 is constructive in the sense that if the inequalities (18) and (19) are satisfied then the closed loop system (11), (12) is locally exponentially stable with an observer gain satisfying (19) and a switching hyperplane given by $\Gamma = P_1 B$ with P_1 satisfying (18). Inequalities (18) and (19) can be easily converted to classical LMI design conditions [4]: there exist $Q_1 \succ 0$, $P_2 \succ 0$, $\mu > 0$, $\theta > 0$ such that

$$Q_1 A^T + AQ_1 - \theta BB^T \preceq -2\alpha Q_1, \tag{42}$$

$$A^T P_2 + P_2 A - \mu C^T C \preceq -2\alpha P_2, \tag{43}$$

with $Q_1 = P_1^{-1}$. Then the matrix defining the switching hyperplanes is given by $\Gamma = Q_1^{-1}B$, and the observer gain is given by $L = -\frac{\mu}{2}CP_2^{-1}$.

Remark 2: Some results in the literature extend the classical separation principle to the case of linear system stabilized via an observer-based nonlinear controller [26]. However, only the case of continuous nonlinear controllers is considered. This property is not verified in the case of relay feedback controller. Note that conditions (42), (43) are independent i.e. they do not have cross variables. Furthermore, from (41) one may notice that when (42) and (43) are satisfied there always exist λ and α such that (38) holds, that is the closed-loop system (18), (19) is stable. This means that the problems of observer synthesis and design of switching surfaces can be addressed independently. It represents a useful extension of the separation principle for systems with relays.

IV. Numerical example

Consider the linear system (4) with

$$u \in \mathcal{V} = \{-v, v\} = \{-5, 5\}, \tag{44}$$

and matrices

$$A = \begin{bmatrix} -1.6 & 1.7 \\ 1.5 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1, 0 \end{bmatrix}. \tag{45}$$

The eigenvalues of A are -2.2, and 2.6 thus the open-loop linear system is unstable. Considering a decay rate $\alpha = 5.5$ an observer based relay feedback controller is designed to stabilize the system to the origin.

After the implementation of the set of LMIs (42)-(43), we find that they are feasible for

$$\begin{aligned}
\theta = 486.8634, \ Q_1 &= \begin{bmatrix} 1.4334 & -4.7050 \\ -4.7050 & 28.6001 \end{bmatrix}, \\
\mu = 348.4742, \ P_2 &= \begin{bmatrix} 37.3918 & -5.4278 \\ -5.4278 & 1.0098 \end{bmatrix}.
\end{aligned} \tag{46}$$

Then, we compute the observer gain

$$L = \begin{bmatrix} -21.2 \\ -113.98 \end{bmatrix}, \tag{47}$$

and the matrix characterizing the switching hyperplanes

$$\Gamma = \begin{bmatrix} 0.25 \\ 0.076 \end{bmatrix}. \tag{48}$$

The computer simulations are performed for the initial conditions $x(0) = \begin{bmatrix} 1, & 0.5 \end{bmatrix}^T$, and $\hat{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T (\xi^T = \begin{bmatrix} 0 & 0 & -1 & -0.5 \end{bmatrix}^T)$ and the results are reported in Figures 1-5.

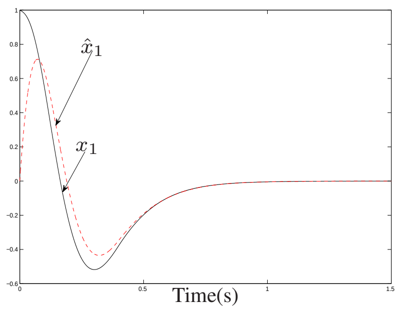

Fig. 1. Real state x_1 and its estimate $\hat{x}_1$

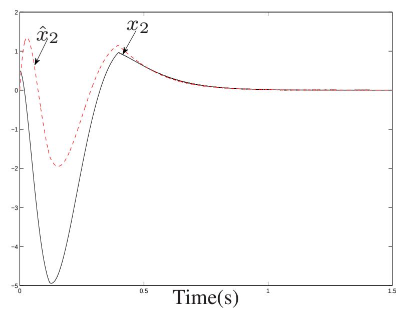

Fig. 2. Real state x_2 and its estimate $\hat{x}_2$

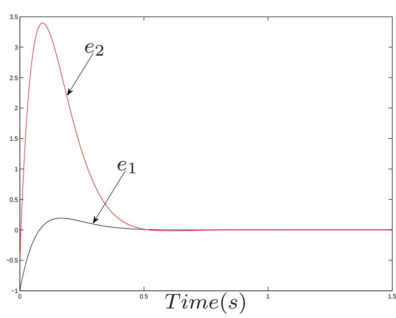

Fig. 3. Observation errors $e = \hat{x} - x$

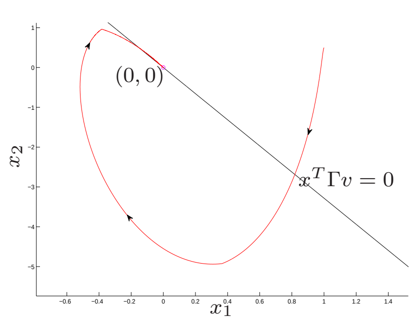

Fig. 4. x_1 and x_2 in the phase plot

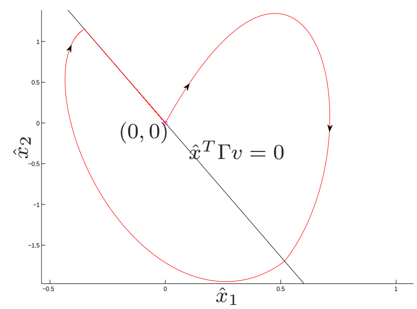

Fig. 5. $\hat{x}_1$ and $\hat{x}_2$ in the phase plot

As we can see from Figures 1 and 2, the states are exactly estimated and they converge to the origin and remain therein. From Figure 3, we can remark that the observation errors converge to zero exponentially, and then the estimated states converge to the real states. In Figure 5 the observer's phase portrait is presented together with the switching hyperplane $\hat{x}^T \Gamma v = 0$. We can observe that the trajectory initialized at zero evolves until reaching the switching hyperplane and it slides over it. The hyperplane $x^T \Gamma v = 0$ and the phase plot of the closed loop system (11), (12) are presented in Figure 4. Comparing Figure 4 and Figure 5, we can see that the hyperplane $x^T \Gamma v = 0$ doesn't coincide exactly with the hyperplane $\hat{x}^T \Gamma v = 0$. This is due to the fact that $\hat{x}$ converges to x when t tends to infinity. In simulations, the trajectory of the closed loop system reaches first the hyperplane $\hat{x}^T \Gamma v = 0$ which tends to $x^T \Gamma v = 0$ as t goes to infinity and slides over it until reaching the origin. Note that, the quantitative estimation of the domain of attraction is not given in this paper. The problem of estimation and the optimization of the domain of attraction will be considered in future works.

V. Conclusion

In this article, the problem of designing an observer-based relay feedback controller for LTI systems is addressed. The control takes values from a finite set of constant vectors and the switching law depends on the estimated states. Qualitative conditions for the stability of the coupled plant are given. The assessment of the performance of the proposed observer-based relay feedback control algorithm is based on the simulation results. The problems of the estimation of the domain of attraction and the study of robustness with respect to perturbations will be studied in depth in future works.

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